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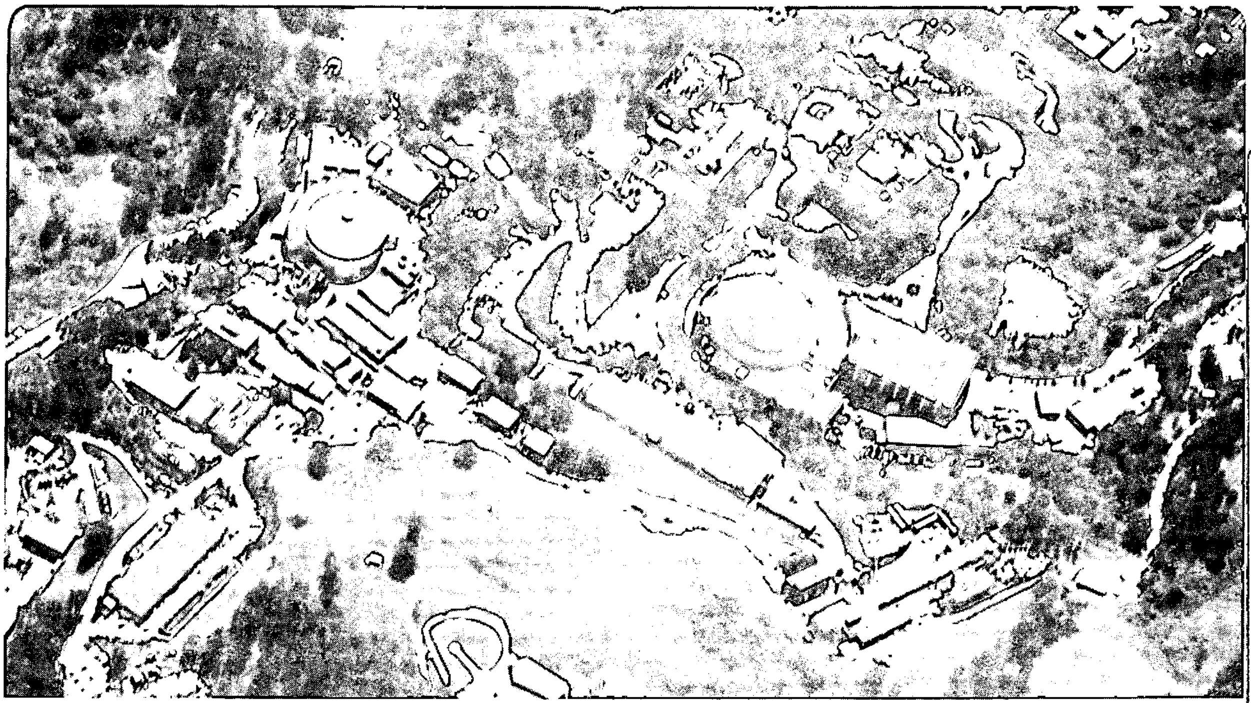

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ROUGH SURFACES AND DIRECTED PERCOLATION[1]

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A recent puzzle in the field of rough surfaces has been to understand the experimental observations of self-affine surfaces that differ from the usual KPZ predictions [4,6]. Anomalous roughening, as it is called, has been seen in a diverse range of experiments including viscous flow and wetting in porous media, the burning of paper, and the growth of bacterial colonies [3,9,11,12].

Self-affine interfaces are characterized by the scaling of the root-mean-square surface width

$$w(l,t) \equiv \langle [h(x,t) - \langle h(x,t)\rangle]^2\rangle^{1/2}. \tag{1}$$

Here $h(x,t)$ is the surface height at time t, and the angular brackets denote the average over x belonging to an interval of size l. It is expected that the width $w(l,t)$ follows the scaling law

$$w(l,t) \sim l^\alpha f(t/l^{\alpha/\beta}). \tag{2}$$

This law was verified for the KPZ equation, in $d = 1+1$, with $\alpha = 1/2$ and $\beta = 1/3$.

Recently, Buldyrev *et al.* [3] have performed imbibition experiments, using paper as the random medium and various aqueous suspensions as wetting fluids. They observed that, after some hours, the interface stopped moving, and they measured a roughness exponent of $\alpha = 0.63 \pm 0.04$ for this stationary interface. To explain this result, they introduced a new model of interfacial growth [2,3]. It incorporates the quenched disorder, pinning and spreading properties of percolation with a rule that erodes overhangs. It was suggested that directed percolation described a new universality class of interfacial growth, and a roughness exponent of $\alpha \simeq 0.63$ was predicted. A similar model giving the same predictions was also advanced by Tang and Leschhorn [10].

In this paper, we present an overview of our numerical and theoretical studies of the static and dynamic properties of this interface model. We map this surface growth model to a network of diodes and resistors and to the geometry of directed percolation. In particular, we demonstrate that the interface is equivalent to the hull (external perimeter) of a critical conductive backbone in the directed percolation problem. This mapping enables us to calculate the fractal dimension and spatial distribution of surface growth sites. Also, we can explain the dynamics of the interfacial growth and observations of an anomalously large ($\alpha \approx 0.75$) roughness exponent. Finally, a possible theory of the avalanche exponent is proposed, and we point to unresolved problems in the model.

1. Model

The principal model examined in this paper is the original one of Buldyrev *et al.* [3], which we call Model A.

Model A – Fast Erosion is defined as follows: On a square lattice of edge L, block a fraction q of the sites corresponding to the inhomogeneous nature of the medium. At each time step we choose all cells that are unblocked dry cells and nearest neighbors to the interface. We wet these cells and all cells below them in their columns. The procedure is then iterated. This process never produces overhangs. A variation on the above theme is *Model B – Slow Erosion.* This model differs from model A only in the way that the columns are eroded. In model B, we erode the blocked sites of a column one by one, until an unblocked cell is reached. Unlike model A, transient overhangs *can* be formed.

2.1 Numerical results: Statics

For small values of q, the wetting proceeds unchecked almost everywhere. Above a critical value q_c, the interface is pinned by blocked sites. This stopped interface is identical for models A and B, and a roughness exponent of $\alpha = 0.63 \pm 0.02$ was found. Figure 1 shows the stopped interface; black squares are dry blocked cells, shaded squares are unblocked dry cells, and wet cells are not shown.

2.2 Numerical results: Dynamics

(i) Roughness exponent α. For $q < q_c$, we measure a roughness exponent for the moving interface: $\alpha = 0.75 \pm .05$. A value for the usual dynamic exponent of $\beta = 0.63 \pm .05$ was also found.

(ii) Fractal dust. Near q_c, the interface does not move upward in a uniform manner. In some regions it may get stuck for a time, in other regions it proceeds by steady erosion. This intermittent behavior must be taken into account in explaining the large dynamic α value (see Section 3.2). Two length scales $\xi_{\parallel}$ and $\xi_{\perp}$ can be extracted from the observed dynamics. Close to but below q_c, however, the front advances only in highly localized regions. At any given time, the projection (onto the x axis) of the live sites forms a *fractal dust*, as shown in Figure 1. We analyze this set in several ways. Just below q_c, the average number of growth sites N_g varies with linear scale r like

$$N_g \sim r^{D_g} \quad (r < \xi_{\parallel}) \tag{3}$$

where $D_g \simeq 0.52 \pm 0.03$ is the fractal dimension of the dust. An alternative description of this scaling is provided by the density-density correlation function $g(r)$ over distances r on the x axis. The numerical data suggest that $g(r)$ behaves as

$$g(r) \sim r^{D_g - 1} f(r/\xi_{\parallel}) \tag{4}$$

where $f(u)$ is a scaling function such that

$$f(u) \sim \begin{cases} \text{const} & u < 1 \\ u^{1-D_g} & u > 1. \end{cases} \tag{5}$$

In log-log plots, we see clear the scaling behavior of $g(r)$. A best fit to the non-zero slope there yields -0.45 ± 0.01, i.e. $D_g = 0.55 \pm 0.01$. The related distribution of the linear distances (projected onto the x axis) between growth sites is is directly related to the growth-site dimension. The length distribution follows the scaling form

$$n(l) \sim l^{-(1+D_g)} \phi(l/\xi_{\|}) \tag{6}$$

where we find $D_g = 0.53 \pm 0.04$. To summarize, all three approaches yield similar values for $D_g \simeq 0.53 \pm 0.03$.

(iii) Avalanches. Space-time plot of growth sites. $\xi_{\|}$ and $\xi_{\perp}$ plot. Avalanche size distribution $n(V)$ and $\tau_{aval} = 1.24$.

Our model can be modified so that even when the growth is completely stopped the blocked cells on the interface may still erode, but at an infinitely low rate. With this assumption, we can remove blocked cells at random when the interface is completely stopped. Each removal will produce an avalanche of growth which will die out when the front is again pinned. We study the distribution of avalanche sizes $P(V)$, and find the scaling relation

$$P(V) \sim V^{-\tau_{aval}} F(V/V_0), \tag{7}$$

where V is the number of sites removed in an avalanche, and $V_0 \sim \xi_{\|}\xi_{\perp}$ is the characteristic volume. The probability $P(V)$ of the occurrence of an avalanche of size V is estimated to be the ratio of the number of avalanches of this size to the total number of avalanches observed. The maximum linear extent of the avalanches, in the horizontal and vertical directions, is found to scale with exponents $\nu_{\|} = 1.73 \pm 0.02$ and $\nu_{\perp} = 1.10 \pm 0.02$. We find the value of the exponent τ_{aval} in (7) to be 1.245 ± 0.02.

3.1 Theory: Statics

(i) Mapping to diode-resistor network.

For theoretical convenience we place the model of Reference 3 on a square bond lattice Λ, tilted by 45° from the usual orientation. As before, we block a fraction q of the bonds. Here the analogous wetting rule is the following: (i) at each time step, wet all unblocked dry bonds which are nearest neighbors to the interface, (ii) wet any bonds that are in the downward 'light cones' of newly-wetted bonds.

It is straightforward to map this particular rule onto a network of resistors and diodes, the so-called "reverse" percolation problem [Redner 1983 in Deutscher et al.]. Blocked bonds map to downward-directed diodes, since diodes allow current to flow in only one direction; unblocked bonds map onto resistors. We therefore need two types of diodes: one set directed southwest (SW), the other southeast (SE). Imagine a line of current sources arrayed along the x-axis. At $q = 1$, the entire lower half plane will carry current. As the concentration of resistors increases (q decreases), backflow will carry current upper a certain height. At a critical value q_c, infinite paths of backflow completely cover the upper half plane of Λ.

This transition is, of course, just a re-phrasing of our wetting critical point. An advantage of the electrical picture is that the duality with a diode-insulator problem (directed percolation) can be clearly appreciated. To see this, construct a lattice Λ' whose bonds are the perpendicular bisectors of the bonds in Λ. SW (SE) diodes in Λ are made to correspond to SE (NE) diodes in Λ'. Likewise, resistors in Λ are mapped onto insulators in Λ'. The correspondence between the circuit elements implies that

$$q = p, \tag{8}$$

where p is the probability of a directed bond in the dual (directed percolation) problem. In particular, $q_c = p_c$.

(ii) Dual directed percolation surface = the backbone hull, $\alpha = 0.633$.

In the diode-resistor network, only a continuous front of downward-directed diodes can stop the current flow. Under the dual mapping, this front corresponds to a spanning path of diodes directed from West to East. However, the diode-insulator construction contains an arbitrary element in that the SE (NE) diodes of Λ' could have just as easily been given the opposite orientation of NW (SW). In this case, the stopped surface then corresponds to an East-to-West path of diodes in the dual lattice. This East-West symmetry is also required by the symmetry of the growth rule. The union of all connecting paths is the conductive *backbone*. Therefore, in the dual lattice, it is the external perimeter (or *hull*) of the backbone that permanently blocks the wetting. For $q \geq q_c$ the interface is pinned and is identical to the backbone hull. Exactly at q_c, the backbone and its hull are self-affine fractals [1,7,8]. As in regular percolation, the backbone has the geometry of a beaded string, where multiply-connected blobs are strung together by short strings of singly-connected sites, the so-called *red* sites. The cluster and backbone are both described by the same $\nu_{\|}$ and $\nu_{\perp}$ since there are only two scaling lengths in the problem. Moreover, the cluster and backbone hulls also share these exponents. Figure 2 shows the conductive backbone on a triangular site lattice. The 'beaded string' geometry of the backbone is apparent.

3.2 Theory: Dynamics

(i) α dynamic

We recast the question of a dynamical α exponent as follows: How can one calculate the variance of the moving- interface width when the dynamics of the interface is manifestly non-uniform? A simple ansatz if that $< w^2 >$ be written as the sum fo two contributions: one from the pinned directed backbone segments, the other from the active 'fractal dust' regions. In the pinned case, the squared width scales with the longitudinal length like $w^2 \sim l^{2\alpha}$. For the "dusty" regions, we assume a jump and a finite slope in the interface, i.e. $w^2 \sim l^2$. Combining the two pieces, weighted by the density along the interface, gives

$$< w^2 > \sim 1 \cdot l^{2\alpha} + l^{D_g - 1} \cdot l^2 \sim l^{D_g + 1}. \qquad (9)$$

On the basis of this picture, we then predict a dynamic α of $(D_g + 1)/2 = 0.765$, to be compared with the observed $0.75 \pm .05$. (ii) Dust dimension. Red bonds $D^{\|}_{red} = 0.577$.

We can construct a theory of the growth dimension D_g from the geometry of the directed-percolation backbone. Imagine the interface at q_c, pinned by the backbone hull. Decreasing the concentration of blocked sites by a small fraction affects the pinned surface only if sites of the backbone hull are removed. For instance, deleting a site belonging to the hull of a blob will trigger a growth spurt with duration dependent on the site's exact position. Once the hull is breached at a red site, then the backbone as a whole is eventually wetted, and the interface moves upward until it encounters another backbone surface. The red sites are therefore the scenes of the most dramatic growth spurts. We identify the red sites with the set of growth sites. The Coniglio argument can be applied to find the fractal dimension of the red sites.

$$D^{\|}_{red} = 1/\nu_{\|} = 0.577. \tag{10}$$

It should be noted that the same argument applied to two-dimensional isotropic percolation is not exact [5]. Nonetheless, there it gives an accurate estimate of the true isotropic exponent, and we may expect the same in our directed model.

(iv) Avalanches. Backbone void picture. τ_{aval} explanation.

Although the directed backbone hull is sufficient to describe the pinned interface, the dynamics of the growth requires some details of the internal structure of the directed backbone.

The backbone is actually a nested collection of pinning surfaces, like a complicated onion, and the succession of avalanches can be thought of as unpeeling this onion. The spaces between the backbone paths we label the *voids*. Since the avalanches erode the backbone void by void, we *first* assume that the size distributions of voids and avalanches are identical. The size of a void will scale like the product of longitudinal and transverse lengths, and this will scale like $\xi_{\|}\xi_{\perp}$. In terms of a longitudinal length l, the scaling dimension of an individual void is then $D = 1 + \nu_\perp/\nu_\parallel$. The number of voids is, however, tied to the scaling dimension of the backbone as a whole, $D^{\parallel}_{bb}$. The void size distribution exponent τ_{void} is then given by

$$\tau_{void} = 1 + D^{\parallel}_{bb}/D = 2 - \frac{2\beta}{\nu_\parallel + \nu_\perp} = 1.80, \qquad (11)$$

where β is the order parameter exponent for the directed percolation problem, and 2β is the exponent for the directed percolation backbone problem. However, in the dynamics, the avalanches are chosen with a distinct bias: larger voids will have larger surfaces and these will be eroded with a frequency proportional to their exposed surface. Thus the above calculation underestimates the number of large avalanches. This is corrected by multiplying $s^{-\tau_{void}}$ by the longitudinal factor $l_\parallel = s^{1/(1+\alpha)}$. We then arrive at a final expression for τ_{aval}:

$$\tau_{aval} = 2 - \frac{\nu_\parallel + 2\beta}{\nu_\parallel + \nu_\perp} = 1.19. \qquad (12)$$

This is significantly outside the error bars of our numerical value for τ_{aval}, and leads us to speculate that a key ingredient is still missing in the theory for the dynamics.

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References

1. B. M. Arora, M. Barma, D. Dhar and M. K. Phani, *J. Phys. C: Solid State Phys.* **16** (1983) 2913-21.
2. A.-L. Barabasi, S. V. Buldyrev, S. Havlin, G. Huber, H. E. Stanley, and T. Vicsek, in

R. Jullien, J. Kertesz, P. Meakin, and D. E. Wolf (eds.), *Surface Disordering: Growth, Roughening, and Phase Transitions* (Nova Science, 1992).

3. S. V. Buldyrev, A.-L. Barabasi, F. Caserta, S. Havlin, H. E. Stanley, and T. Vicsek, in *Phys. Rev. A* **46** (1992) R8313.
4. *Dynamics of Fractal Surfaces* F.Family and T. Vicsek (eds.), (World Scientific, Singapore, 1991).
5. S. Havlin and R. Nossal, *J. Phys. A* **17** (1984) L427.
6. M. Kardar, G. Parisi and Y.-C. Zhang, *Phys. Rev. Lett.* **56** (1986) 889.
7. S. Redner and J. S. Brooks, *J. Phys. A: Math. Gen.* **15** (1982) L605-L610.
8. S. Redner, in *Percolation Structures and Processes*, G. Deutscher, R. Zallen and J. Adler (eds.) (Adam Hilger, Bristol, 1982).
9. M. A. Rubio, C. A. Edwards, A. Dougherty and J. P. Gollub, *Phys. Rev. Lett.* **63** (1990) 1685.
10. L.-H. Tang and H. Leschhorn, *Phys. Rev. A* **45** (1992) R8309.
11. T. Vicsek, M. Cserzo and V. K. Horvath, *Physica A* **167** (1990) 315.
12. J. Zhang, Y.-C. Zhang, P. Alstrom and M. T. Levinsen, in R. Jullien, J. Kertesz, P. Meakin, and D. E. Wolf (eds.), *Surface Disordering: Growth, Roughening, and Phase Transitions* (Nova Science, 1992).
13. Using duality arguments specific to certain two-dimensional isotropic lattices, our red-site conjecture is equivalent to identifying the growth dimension $D - D_{min} \approx 0.766$ with $D_{red} = 3/4$. This approximate relation was first observed numerically by Havlin and Nossal [5] in dimension two and greater.

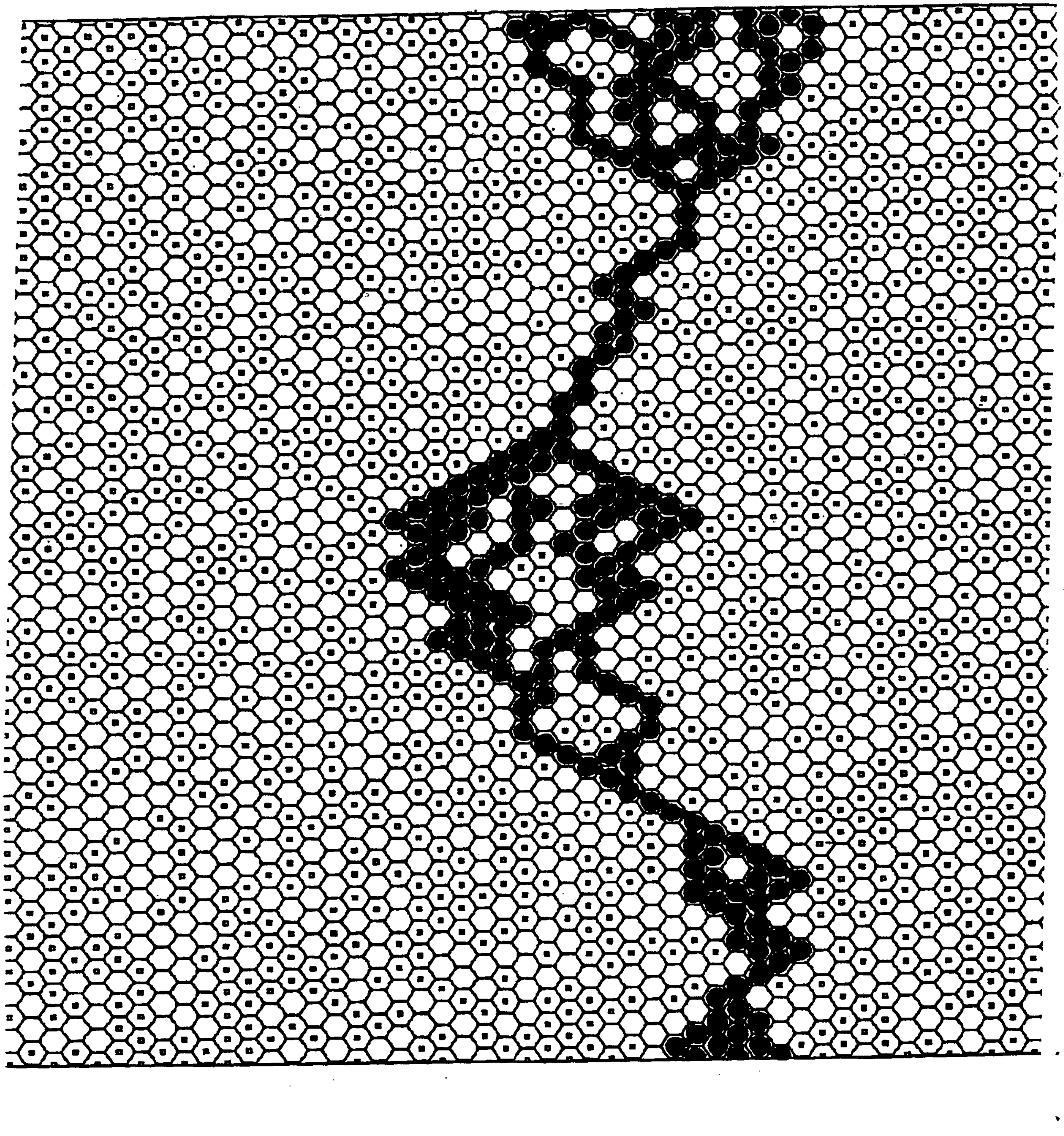
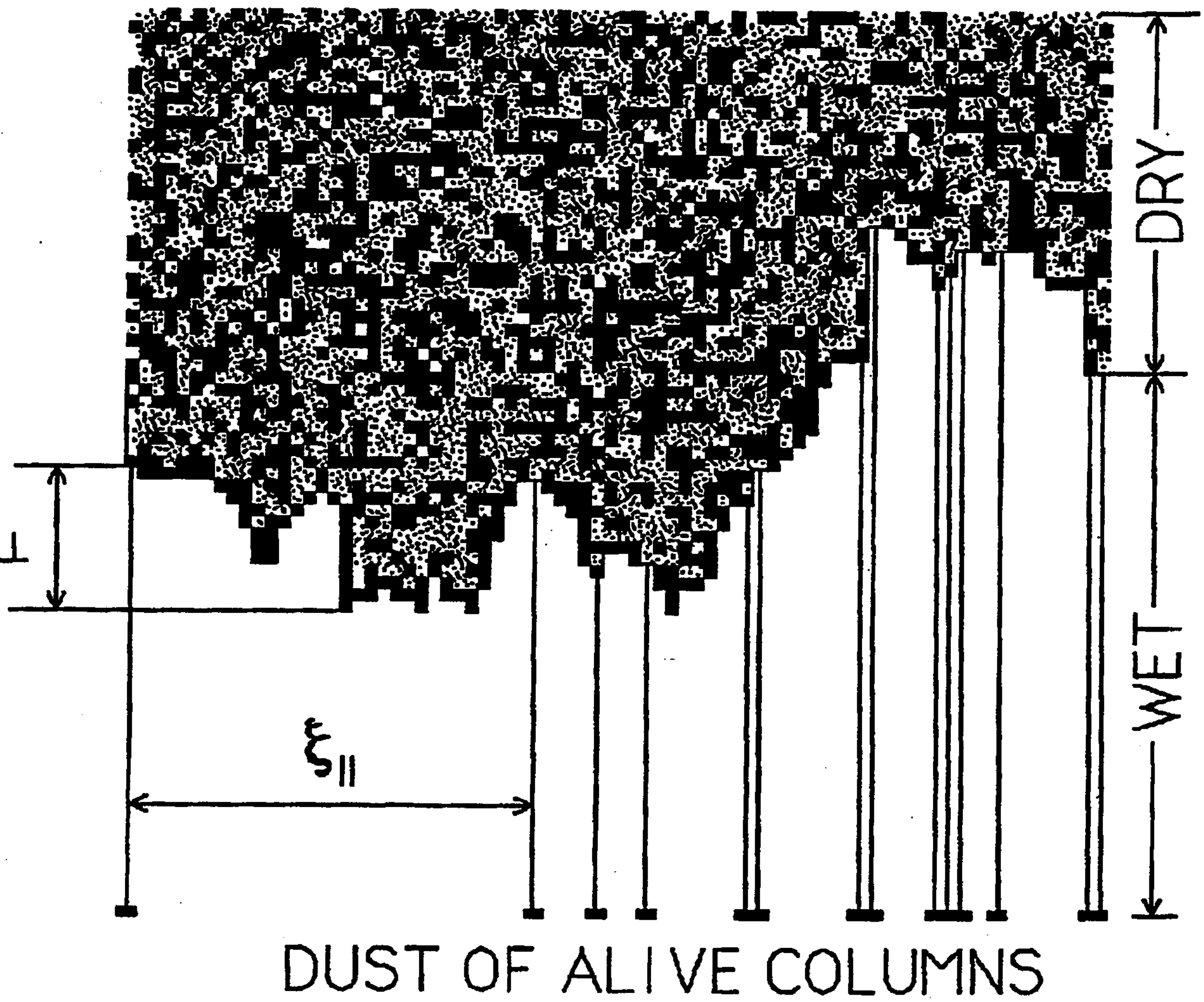


Figure 2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DUST OF ALIVE COLUMNS

Figure 1

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